

# The Square Trees in the Tribonacci Sequence

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**Abstract:** The Tribonacci sequence  $\mathbb{T}$  is the fixed point of the substitution  $\sigma(a, b, c) = (ab, ac, a)$ . In this note, we get the explicit expressions of all squares, and then establish the tree structure of the positions of repeated squares in  $\mathbb{T}$ , called square trees. Using the square trees, we give a fast algorithm for counting the number of repeated squares in  $\mathbb{T}[1, n]$  for all  $n$ , where  $\mathbb{T}[1, n]$  is the prefix of  $\mathbb{T}$  of length  $n$ . Moreover we get explicit expressions for some special  $n$  such as  $n = t_m$  (the Tribonacci number) etc., which including some known results such as H.Mousavi and J.Shallit[6].

**Key words:** the Tribonacci sequence, kernel, square, gap sequence.

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## 1 Introduction

Let  $\mathcal{A} = \{a, b, c\}$  be a three-letter alphabet. The Tribonacci sequence  $\mathbb{T}$  is the fixed point beginning with  $a$  of the substitution  $\sigma(a, b, c) = (ab, ac, a)$ . As a natural generalization of the Fibonacci sequence,  $\mathbb{T}$  has been studied extensively by many authors, see [1, 6, 7, 8].

Let  $\omega$  be a factor of  $\mathbb{T}$ , denoted by  $\omega \prec \mathbb{T}$ . Let  $\omega_p$  be the  $p$ -th occurrence of  $\omega$ . If the factor  $\omega$  and integer  $p$  such that  $\omega_p \omega_{p+1} \prec \mathbb{T}$ , we call  $\omega_p \omega_{p+1}$  a square of  $\mathbb{T}$ . As we know,  $\mathbb{T}$  contains no fourth powers. The properties of squares and cubes are objects of a great interest in mathematics and computer science etc.

We denote by  $|\omega|$  the length of  $\omega$ . Denote  $|\omega|_\alpha$  the number of letter  $\alpha$  in  $\omega$ , where  $\alpha \in \mathcal{A}$ . Let  $\tau = x_1 \cdots x_n$  be a finite word (or  $\tau = x_1 x_2 \cdots$  be a sequence). For any  $i \leq j \leq n$ , define  $\tau[i, j] = x_i x_{i+1} \cdots x_{j-1} x_j$ . By convenient, denote  $\tau[i] = \tau[i, i] = x_i$ ,  $\tau[i, i-1] = \varepsilon$  (empty word). Denote  $T_m = \sigma^m(a)$  for  $m \geq 0$ ,  $T_{-2} = \varepsilon$ ,  $T_{-1} = c$ . Denote  $t_m = |T_m|$  for  $m \geq -2$ , called the  $m$ -th Tribonacci number. Denote by  $\delta_m$  the last letter of  $T_m$  for  $m \geq -1$ .

Denote  $A(n) = \#\{\omega, p : \omega_p \omega_{p+1} \prec \mathbb{T}[1, n]\}$  the number of repeated squares in  $\mathbb{T}[1, n]$ . In 2014, H.Mousavi and J.Shallit[6] gave expression of  $A(t_m)$ , which they proved by mechanical way. In [4], we give a fast algorithm for counting the number of repeated squares in each prefix of the Fibonacci sequence. In this note, we give a fast algorithm for counting  $A(n)$  for all  $n$ . In Section 2, we establish the tree structure of the positions of repeated squares in  $\mathbb{T}$ , called the square trees. Section 3 is devoted to give a fast algorithm for counting  $A(n)$ . As a special case, we get expression of  $A(t_m)$  in Section 4.

The main tools of the paper are “kernel word” and “gap sequence”, which introduced and studied in [2]. We define the kernel numbers that  $k_0 = 0$ ,  $k_1 = k_2 = 1$ ,  $k_m = k_{m-1} + k_{m-2} + k_{m-3} - 1$  for  $m \geq 3$ . The kernel word with order  $m$  is defined as  $K_1 = a$ ,  $K_2 = b$ ,  $K_3 = c$ ,  $K_m = \delta_{m-1} T_{m-3} [1, k_m - 1]$  for  $m \geq 4$ . Using the property of gap sequence, we can determine the positions of all  $\omega_p$ , and then establish the square trees.

## 2 The square trees

In [3], we determined the three cases of squares with kernel  $K_m$  (i.e., the maximal kernel word occurring in these squares is  $K_m$ ). For  $m \geq 4$  and  $p \geq 1$ , we denote

$$\Lambda(m, p) = pt_{m-1} + |\mathbb{T}[1, p-1]|_a (t_{m-2} + t_{m-3}) + |\mathbb{T}[1, p-1]|_b t_{m-2}.$$

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By Property 6.1 in [5], we have the position of the last letter of the  $p$ -th occurrence of  $K_m$  that  $P(K_m, p) = \Lambda(m, p) + k_m - 1$ . Thus we can define three sets for  $m \geq 4$  and  $p \geq 1$ , which contain the three cases of squares, respectively.

$$\begin{cases} \langle 1, K_m, p \rangle &= \{P(\omega\omega, p) : Ker(\omega\omega) = K_m, |\omega| = t_{m-1}, \omega\omega \prec \mathbb{T}\} \\ &= \{\Lambda(m, p) + t_{m-1}, \dots, \Lambda(m, p) + k_{m+3} - 2\}; \\ \langle 2, K_m, p \rangle &= \{P(\omega\omega, p) : Ker(\omega\omega) = K_m, |\omega| = t_{m-4} + t_{m-3}, \omega\omega \prec \mathbb{T}\} \\ &= \{\Lambda(m, p) + t_{m-3} + t_{m-4}, \dots, \Lambda(m, p) + k_{m+2} - 2\}; \\ \langle 3, K_m, p \rangle &= \{P(\omega\omega, p) : Ker(\omega\omega) = K_m, |\omega| = t_{m-4} - k_{m-3}, \omega\omega \prec \mathbb{T}\} \\ &= \{\Lambda(m, p) + k_m - 1, \dots, \Lambda(m, p) + 2t_{m-4} - 1\}. \end{cases}$$

For  $m \geq 4$  and  $p \geq 1$ , we consider the sets

$$\begin{cases} \Gamma_{1,m,p} = \{\Lambda(m, p) + k_{m+2} - 1, \dots, \Lambda(m, p) + k_{m+3} - 2\}; \\ \Gamma_{2,m,p} = \{\Lambda(m, p) + k_{m+1} - 1, \dots, \Lambda(m, p) + k_{m+2} - 2\}; \\ \Gamma_{3,m,p} = \{\Lambda(m, p) + k_m - 1, \dots, \Lambda(m, p) + k_{m+1} - 2\}. \end{cases}$$

Obviously,  $\langle 1, K_m, p \rangle$  (resp.  $\langle 2, K_m, p \rangle$ ,  $\langle 3, K_m, p \rangle$ ) contains the several maximal (resp. maximal, minimal) elements of  $\Gamma_{1,m,p}$  (resp.  $\Gamma_{2,m,p}$ ,  $\Gamma_{3,m,p}$ ). Moreover  $\max \Gamma_{2,m,p} + 1 = \min \Gamma_{1,m,p}$  and  $\max \Gamma_{3,m,p} + 1 = \min \Gamma_{2,m,p}$ . Using Lemma 6.4 in [5], comparing minimal and maximal elements in these sets below, we have

$$\begin{cases} \Gamma_{1,m,p} = \Gamma_{3,m-1,P(a,p)+1} \cup \Gamma_{2,m-1,P(a,p)+1} \cup \Gamma_{1,m-1,P(a,p)+1}, & m \geq 4; \\ \Gamma_{2,m,p} = \Gamma_{3,m-2,P(b,p)+1} \cup \Gamma_{2,m-2,P(b,p)+1} \cup \Gamma_{1,m-2,P(b,p)+1}, & m \geq 5; \\ \Gamma_{3,m,p} = \Gamma_{3,m-3,P(c,p)+1} \cup \Gamma_{2,m-3,P(c,p)+1} \cup \Gamma_{1,m-3,P(c,p)+1}, & m \geq 6. \end{cases}$$

Thus we establish the recursive relations for any  $\Gamma_{1,m,p}$  ( $m \geq 4$ ),  $\Gamma_{2,m,p}$  ( $m \geq 5$ ) and  $\Gamma_{3,m,p}$  ( $m \geq 6$ ). By the relation between  $\Gamma_{i,m,p}$  and  $\langle i, K_m, p \rangle$ , we get the tree structure of the positions of repeated squares in  $\mathbb{T}$ , called the square trees.

$$\begin{cases} \pi_1 \langle 1, K_m, p \rangle = \langle 3, K_{m-1}, \hat{a} \rangle \cup \langle 2, K_{m-1}, \hat{a} \rangle \cup \langle 1, K_{m-1}, \hat{a} \rangle, m \geq 4; \\ \pi_2 \langle 2, K_m, p \rangle = \langle 3, K_{m-2}, \hat{b} \rangle \cup \langle 2, K_{m-2}, \hat{b} \rangle \cup \langle 1, K_{m-2}, \hat{b} \rangle, m \geq 5; \\ \pi_3 \langle 3, K_m, p \rangle = \langle 3, K_{m-3}, \hat{c} \rangle \cup \langle 2, K_{m-3}, \hat{c} \rangle \cup \langle 1, K_{m-3}, \hat{c} \rangle, m \geq 6. \end{cases}$$

Here we denote  $P(\alpha, p) + 1$  by  $\hat{\alpha}$  for short,  $\alpha \in \{a, b, c\}$ .

On the other hand, for  $m \geq 4$  and  $i \in \{1, 2, 3\}$ , each  $\langle i, K_m, 1 \rangle$  belongs to the square trees. Moreover  $\langle i, K_m, \hat{a} \rangle$  (resp.  $\langle i, K_m, \hat{b} \rangle$ ,  $\langle i, K_m, \hat{c} \rangle$ ) is subset of  $\pi_1 \langle 1, K_{m+1}, p \rangle$  (resp.  $\pi_2 \langle 2, K_{m+2}, p \rangle$ ,  $\pi_3 \langle 3, K_{m+3}, p \rangle$ ). Notice that  $\mathbb{N} = \{1\} \cup \{P(a, p) + 1\} \cup \{P(b, p) + 1\} \cup \{P(c, p) + 1\}$ , the square trees contain all  $\langle i, K_m, p \rangle$ , i.e. all squares in  $\mathbb{T}$ . Fig. 1 shows some examples.

### 3 Algorithm: the numbers of repeated squares in $\mathbb{T}[1, n]$

Denote  $a(n) = \#\{(\omega, p) : \omega_p \omega_{p+1} \triangleright \mathbb{T}[1, n]\}$  the number of squares ending at position  $n$ . By Proposition 3.1, we can calculate  $a(n)$ , and obversely calculate  $A(n)$  by  $A(n) = \sum_{i=4}^n a(i)$ . For  $m \geq 4$ , since  $k_m = \frac{t_m - 2t_{m-1} + t_{m-2} + 1}{2}$ ,

$$\begin{cases} \Gamma_{1,m,1} = \left\{ \frac{t_m + 2t_{m-1} - t_{m-2} - 1}{2}, \dots, \frac{t_m + 2t_{m-1} + t_{m-2} - 3}{2} \right\}; \\ \Gamma_{2,m,1} = \left\{ \frac{-t_m + 4t_{m-1} + t_{m-2} - 1}{2}, \dots, \frac{t_m + 2t_{m-1} - t_{m-2} - 3}{2} \right\}; \\ \Gamma_{3,m,1} = \left\{ \frac{t_m + t_{m-2} - 1}{2}, \dots, \frac{-t_m + 4t_{m-1} + t_{m-2} - 3}{2} \right\}. \end{cases}$$

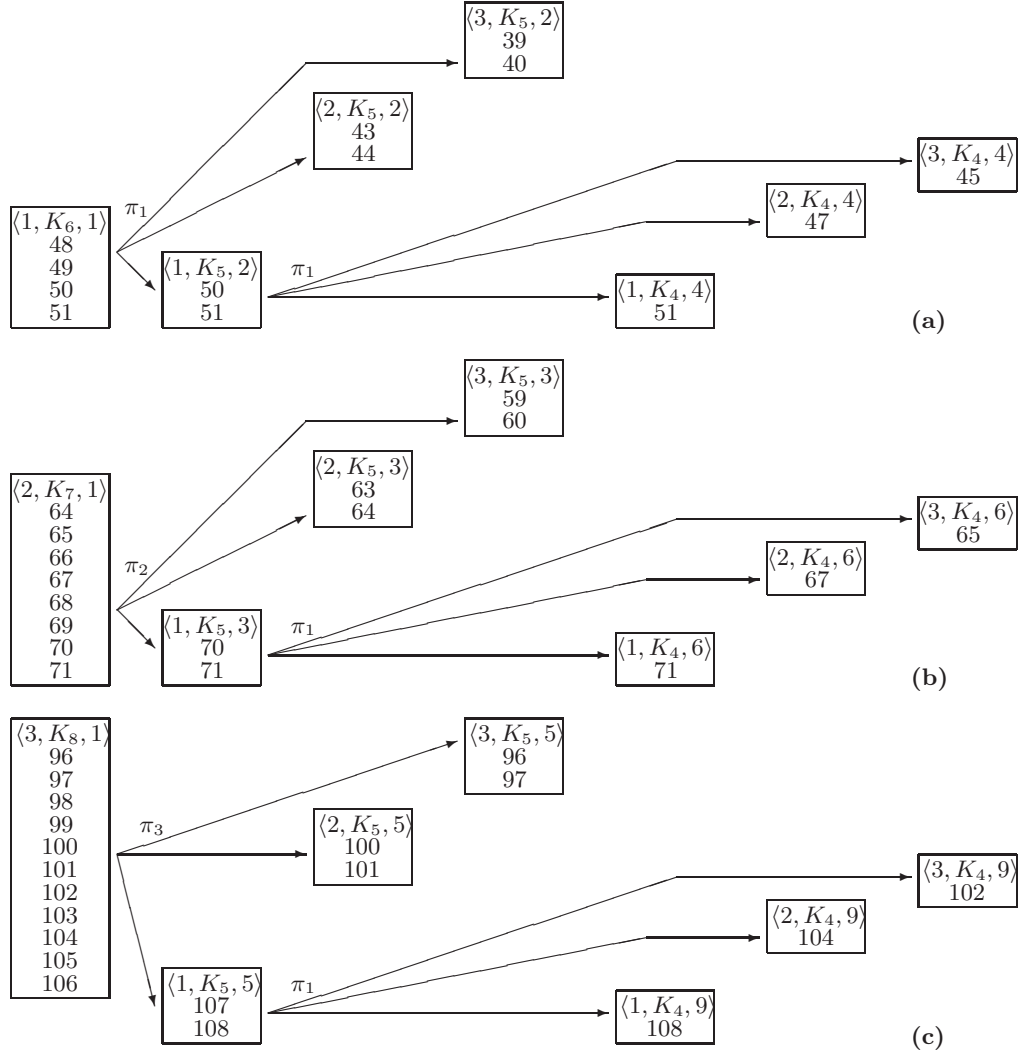


Figure 1: (a)-(c) are square trees from root  $\langle 1, K_6, 1 \rangle$ ,  $\langle 2, K_7, 1 \rangle$ ,  $\langle 3, K_7, 1 \rangle$ , respectively.

- (a)  $\Phi_m = \frac{m}{22}(-5t_m + 14t_{m-1} + 4t_{m-2}) + \frac{1}{44}(67t_m - 166t_{m-1} + 5t_{m-2}) + \frac{1}{4}$ .  
(b)  $\sum a(\Gamma_{i,4,1}) = 1$  where  $i \in \{1, 2, 3\}$ , and

$$\begin{cases} \sum a(\Gamma_{1,m,1}) = \frac{m}{22}(4t_m - 9t_{m-1} + 10t_{m-2}) + \frac{1}{44}(19t_m + 36t_{m-1} - 169t_{m-2}) - \frac{1}{4}; \\ \sum a(\Gamma_{2,m,1}) = \frac{m}{22}(10t_m - 6t_{m-1} - 19t_{m-2}) + \frac{1}{44}(-189t_m + 156t_{m-1} + 331t_{m-2}) - \frac{1}{4}; \\ \sum a(\Gamma_{3,m,1}) = \frac{m}{22}(-19t_m + 29t_{m-1} + 13t_{m-2}) + \frac{1}{44}(237t_m - 358t_{m-1} - 157t_{m-2}) + \frac{3}{4}. \end{cases}$$

- (c)  $\sum_{j=4}^{m-1} \Phi_j = \frac{m}{44}(13t_m - 10t_{m-1} + 5t_{m-2}) + \frac{2}{11}(-8t_m + 8t_{m-1} - 7t_{m-2}) + \frac{m}{4} + 2$ .

$$(d) \begin{cases} A(\max \Gamma_{3,m,1}) = \frac{m}{44}(-25t_m + 48t_{m-1} + 31t_{m-2}) + \frac{1}{44}(173t_m - 294t_{m-1} - 213t_{m-2}) + \frac{m+11}{4}; \\ A(\max \Gamma_{2,m,1}) = \frac{m}{44}(-5t_m + 36t_{m-1} - 7t_{m-2}) + \frac{1}{22}(-8t_m - 69t_{m-1} + 59t_{m-2}) + \frac{m+10}{4}; \\ A(\max \Gamma_{1,m,1}) = \frac{m}{44}(3t_m + 18t_{m-1} + 13t_{m-2}) + \frac{1}{44}(3t_m - 102t_{m-1} - 51t_{m-2}) + \frac{m+9}{4}. \end{cases}$$

Figure 2: These properties can be proved easily by induction, where (a) and (c) hold for  $m \geq 4$ , (b) and (d) hold for  $m \geq 5$ .

**Proposition 3.1.**  $a([8]) = [1]$ ,  $a([9, 10]) = [0, 1]$ ,  $a([11, \dots, 14]) = [0, 0, 0, 1]$ ,  $a([15, 16]) = [1, 1]$ ,  $a([17, \dots, 20]) = [0, 0, 1, 1]$ ,  $a([28, \dots, 31]) = [1, 1, 1, 1]$ ,

$$\begin{cases} a(\Gamma_{1,m,1}) = [a(\Gamma_{3,m-1,1}), a(\Gamma_{2,m-1,1}), a(\Gamma_{1,m-1,1})] + [\underbrace{0, \dots, 0}_{t_{m-2}-k_m+1}, \underbrace{1, \dots, 1}_{k_m-1}]; \\ a(\Gamma_{2,m,1}) = [a(\Gamma_{3,m-2,1}), a(\Gamma_{2,m-2,1}), a(\Gamma_{1,m-2,1})] + [\underbrace{0, \dots, 0}_{t_{m-3}-k_m+1}, \underbrace{1, \dots, 1}_{k_m-1}]; \\ a(\Gamma_{3,m,1}) = [a(\Gamma_{3,m-3,1}), a(\Gamma_{2,m-3,1}), a(\Gamma_{1,m-3,1})] + [\underbrace{1, \dots, 1}_{t_{m-4}-k_{m-3}+1}, \underbrace{0, \dots, 0}_{k_{m-3}-1}]. \end{cases}$$

Denote  $\Phi_m = \sum a(\Gamma_{3,m,1}) + \sum a(\Gamma_{2,m,1}) + \sum a(\Gamma_{1,m,1})$ . The immediately corollaries of Proposition 3.1 are  $\sum a(\Gamma_{1,m,1}) = \Phi_{m-1} + k_m - 1$ ,  $\sum a(\Gamma_{2,m,1}) = \Phi_{m-2} + k_m - 1$ ,  $\sum a(\Gamma_{3,m,1}) = \Phi_{m-3} + t_{m-4} - k_{m-3} + 1$ . Moreover, for  $m \geq 7$ ,

$$\Phi_m = \Phi_{m-1} + \Phi_{m-2} + \Phi_{m-3} + \frac{-3t_m + 6t_{m-1} + t_{m-2} - 1}{2}.$$

By induction, we can prove the 4 properties in Fig 2.

Obversely we can calculate  $A(n)$  by  $A(n) = \sum_{i=4}^n a(i)$ . But when  $n$  large, this method is complicated. Now we turn to give a fast algorithm. For any  $n \geq 52$ , let  $m$  such that  $n \in \Gamma_{3,m,1} \cup \Gamma_{2,m,1} \cup \Gamma_{1,m,1} = \{\frac{t_m+t_{m-2}-1}{2}, \dots, \frac{t_m+2t_{m-1}+t_{m-2}-3}{2}\}$ . We already determine the expression of  $A(\max \Gamma_{i,m,1})$  for  $i \in \{1, 2, 3\}$ ,  $m \geq 5$ . In order to calculate  $A(n)$ , we only need to calculate  $\sum_{i=\min \Gamma_{i,m,1}}^n a(i)$ .

**Algorithm.** Step 1. For  $n \leq 51$ , calculate  $\sum_{i=\min \Gamma_{i,m,1}}^n a(i)$  by Property 3.1.

Step 2. For  $n \geq 52$ , find the  $m$  and  $i$  such that  $n \in \Gamma_{i,m,1}$ , then  $m \geq 7$ . We calculate  $\sum_{i=\min \Gamma_{i,m,1}}^n a(i)$  by the properties in Fig.3.

Step 3. Calculate  $A(\min \Gamma_{i,m,1} - 1)$  by the Property (d) in Fig.2.

Step 4.  $A(n) = A(\min \Gamma_{i,m,1} - 1) + \sum_{i=\min \Gamma_{i,m,1}}^n a(i)$ .

## 4 Expression: the numbers of repeated squares in $T_m$

Since  $\theta_m^8 \leq t_m < \theta_m^9$  and  $\theta_{m-1}^6 \leq t_m - t_{m-1} < \theta_{m-1}^7$  for  $m \geq 7$ , see Fig.3,

$$\begin{aligned} & \sum_{i=\min \Gamma_{1,m,1}}^{t_m} a(i) - \sum_{i=\min \Gamma_{1,m-3,1}}^{t_{m-3}} a(i) \\ &= \sum a(\Gamma_{3,m-1,1}) + \sum a(\Gamma_{3,m-3,1}) + \sum a(\Gamma_{2,m-3,1}) + 2t_m - 2t_{m-1} - 3t_{m-2} + 1 \\ &= \frac{m}{22}(-19t_m + 29t_{m-1} + 13t_{m-2}) + \frac{1}{44}(347t_m - 622t_{m-1} - 47t_{m-2}) + \frac{9}{4}. \end{aligned}$$

(a)  $n \in \Gamma_{3,m,1} = \left\{ \frac{t_m+t_{m-2}-1}{2}, \dots, \frac{-t_m+4t_{m-1}+t_{m-2}-3}{2} \right\}$  for  $m \geq 7$ . Denote

$$\begin{cases} \theta_m^1 = \min \Gamma_{3,m,1} = \frac{t_m+t_{m-2}-1}{2}; \\ \theta_m^2 = \min \Gamma_{3,m,1} + |\Gamma_{3,m-3,1}| = \frac{-5t_m+10t_{m-1}+3t_{m-2}-1}{2}; \\ \theta_m^3 = \min \Gamma_{3,m,1} + |\Gamma_{3,m-3,1}| + |\Gamma_{2,m-3,1}| = \frac{-t_m+6t_{m-1}-3t_{m-2}-1}{2}; \\ \eta_m^1 = \min \Gamma_{3,m,1} + t_{m-4} - k_{m-3} + 1 = -2t_m + 5t_{m-1}. \\ \theta_m^4 = \max \Gamma_{3,m,1} + 1 = \min \Gamma_{2,m,1} = \frac{-t_m+4t_{m-1}+t_{m-2}-1}{2}. \end{cases}$$

Obviously,  $\theta_m^3 < \eta_m^1 < \theta_m^4$  for  $m \geq 7$ , and  $\min \Gamma_{3,m,1} - \min \Gamma_{3,m-3,1} = t_{m-1}$ . By Property 3.1, we have: for  $n \geq 52$ , let  $m$  such that  $n \in \Gamma_{3,m,1}$ , then  $m \geq 7$  and

$$\begin{aligned} & \sum_{i=\min \Gamma_{3,m,1}}^n a(i) \\ &= \begin{cases} \sum_{i=\min \Gamma_{3,m-3,1}}^{n-t_{m-1}} a(i) + n - \min \Gamma_{3,m,1} + 1, & \theta_m^1 \leq n < \theta_m^2; \\ \sum_{i=\min \Gamma_{2,m-3,1}}^{n-t_{m-1}} a(i) + \sum a(\Gamma_{3,m-3,1}) + n - \min \Gamma_{3,m,1} + 1, & \theta_m^2 \leq n < \theta_m^3; \\ \sum_{i=\min \Gamma_{1,m-3,1}}^{n-t_{m-1}} a(i) + \sum a(\Gamma_{3,m-3,1}) + \sum a(\Gamma_{2,m-3,1}) + n - \min \Gamma_{3,m,1} + 1, & \theta_m^3 \leq n < \eta_m^1; \\ \sum_{i=\min \Gamma_{1,m-3,1}}^{n-t_{m-1}} a(i) + \sum a(\Gamma_{3,m-3,1}) + \sum a(\Gamma_{2,m-3,1}) + \frac{-5t_m+10t_{m-1}-t_{m-2}+1}{2}, & \text{otherwise.} \end{cases} \end{aligned}$$

(b)  $n \in \Gamma_{2,m,1} = \left\{ \frac{-t_m+4t_{m-1}+t_{m-2}-1}{2}, \dots, \frac{t_m+2t_{m-1}-t_{m-2}-3}{2} \right\}$  for  $m \geq 6$ . Denote

$$\begin{cases} \theta_m^5 = \min \Gamma_{2,m,1} + |\Gamma_{3,m-2,1}| = \frac{3t_m-5t_{m-2}-1}{2}; \\ \eta_m^2 = \min \Gamma_{2,m,1} + t_{m-3} - k_m + 1 = 2t_{m-1} - t_{m-2}. \\ \theta_m^6 = \min \Gamma_{3,m,1} + |\Gamma_{3,m-3,1}| + |\Gamma_{2,m-3,1}| = \frac{3t_m-2t_{m-1}-t_{m-2}-1}{2}; \\ \theta_m^7 = \max \Gamma_{2,m,1} + 1 = \min \Gamma_{1,m,1} = \frac{t_m+2t_{m-1}-t_{m-2}-1}{2}. \end{cases}$$

Obviously,  $\theta_m^5 < \eta_m^2 \leq \theta_m^6$  for  $m \geq 6$ , and  $\min \Gamma_{2,m,1} - \min \Gamma_{3,m-2,1} = t_{m-1}$ . By Property 3.1, we have: for  $n \geq 32$ , let  $m$  such that  $n \in \Gamma_{2,m,1}$ , then  $m \geq 6$  and

$$\begin{aligned} & \sum_{i=\min \Gamma_{2,m,1}}^n a(i) \\ &= \begin{cases} \sum_{i=\min \Gamma_{3,m-2,1}}^{n-t_{m-1}} a(i), & \theta_m^4 \leq n < \theta_m^5; \\ \sum_{i=\min \Gamma_{2,m-2,1}}^{n-t_{m-1}} a(i) + \sum a(\Gamma_{3,m-2,1}), & \theta_m^5 \leq n < \eta_m^2; \\ \sum_{i=\min \Gamma_{2,m-2,1}}^{n-t_{m-1}} a(i) + \sum a(\Gamma_{3,m-2,1}) + n - \eta_m^2 + 1, & \eta_m^2 \leq n < \theta_m^6; \\ \sum_{i=\min \Gamma_{1,m-2,1}}^{n-t_{m-1}} a(i) + \sum a(\Gamma_{3,m-2,1}) + \sum a(\Gamma_{2,m-2,1}) + n - \eta_m^2 + 1, & \text{otherwise.} \end{cases} \end{aligned}$$

(c)  $n \in \Gamma_{1,m,1} = \left\{ \frac{t_m+2t_{m-1}-t_{m-2}-1}{2}, \dots, \frac{t_m+2t_{m-1}+t_{m-2}-3}{2} \right\}$  for  $m \geq 5$ . Denote

$$\begin{cases} \theta_m^8 = \min \Gamma_{1,m,1} + |\Gamma_{3,m-1,1}| = \frac{t_m+3t_{m-2}-1}{2}; \\ \theta_m^9 = \min \Gamma_{1,m,1} + |\Gamma_{3,m-1,1}| + |\Gamma_{2,m-1,1}| = \frac{-t_m+4t_{m-1}+3t_{m-2}-1}{2}; \\ \eta_m^3 = \min \Gamma_{1,m,1} + t_{m-2} - k_m + 1 = 2t_{m-1}. \\ \theta_m^{10} = \max \Gamma_{1,m,1} + 1 = \min \Gamma_{3,m+1,1} = \frac{t_m+2t_{m-1}+t_{m-2}-1}{2}. \end{cases}$$

Obviously,  $\theta_m^9 < \eta_m^3 < \theta_m^{10}$  for  $m \geq 5$ , and  $\min \Gamma_{1,m,1} - \min \Gamma_{3,m-1,1} = t_{m-1}$ . By Property 3.1, we have: for  $n \geq 21$ , let  $m$  such that  $n \in \Gamma_{1,m,1}$ , then  $m \geq 5$  and

$$\begin{aligned} & \sum_{i=\min \Gamma_{1,m,1}}^n a(i) \\ &= \begin{cases} \sum_{i=\min \Gamma_{3,m-1,1}}^{n-t_{m-1}} a(i), & \theta_m^7 \leq n < \theta_m^8; \\ \sum_{i=\min \Gamma_{2,m-1,1}}^{n-t_{m-1}} a(i) + \sum a(\Gamma_{3,m-1,1}), & \theta_m^8 \leq n < \theta_m^9; \\ \sum_{i=\min \Gamma_{1,m-1,1}}^{n-t_{m-1}} a(i) + \sum a(\Gamma_{3,m-1,1}) + \sum a(\Gamma_{2,m-1,1}), & \theta_m^9 \leq n < \eta_m^3; \\ \sum_{i=\min \Gamma_{1,m-1,1}}^{n-t_{m-1}} a(i) + \sum a(\Gamma_{3,m-1,1}) + \sum a(\Gamma_{2,m-1,1}) + n - \eta_m^3 + 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Figure 3: (a)-(c) show the three cases of recursive relations between  $\sum_{i=\min \Gamma_{k,m,1}}^n a(i)$  and  $\sum_{i=\min \Gamma_{t,m-k,1}}^n a(i)$ , where  $k, t \in \{1, 2, 3\}$ , respectively. These relations are derived directly from the square trees (the tree structure of the positions of repeated squares). Using them, we can calculate  $\sum_{i=\min \Gamma_{k,m,1}}^n a(i)$  fast, and give a fast algorithm for  $A(n)$ .

For  $m \geq 7$ , by induction,  $\sum_{i=\min \Gamma_{1,m,1}}^{t_m} a(i)$  is equal to

$$\frac{m}{44}(23t_m - 38t_{m-1} - 3t_{m-2}) + \frac{1}{44}(-65t_m + 164t_{m-1} - 105t_{m-2}) + \frac{3m}{4} - \frac{9}{4}.$$

Since  $\min \Gamma_{1,m,1} - 1 = \max \Gamma_{2,m,1}$ ,  $A(t_m) = A(\max \Gamma_{2,m,1}) + \sum_{i=\min \Gamma_{1,m,1}}^{t_m} a(i)$ . By the properties in Fig.3, we can prove Theorem 21 in in H.Mousavi and J.Shallit[6] in a novel way: for  $m \geq 3$ ,

$$A(t_m) = \frac{m}{22}(9t_m - t_{m-1} - 5t_{m-2}) + \frac{1}{44}(-81t_m + 26t_{m-1} + 13t_{m-2}) + m + \frac{1}{4}.$$

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